Orientation Order Analysis of Spherical Four-Bar Mechanisms

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In this paper, we present a novel methodology for orientation order analysis of spherical RR dyads. The methodology is a spherical generalization of the recent works of Myszka, Murray, and Schmiedeler for assessing position order of planar RR dyads. The objective of the methodology is to determine if a prescribed fixed axis location for a spherical RR dyad will result in the dyad guiding a moving rigid-body through a set of finitely separated spherical orientations in the desired order (e.g., 1, 2, 3, 4, etc.). The planar propeller methodology of Myszka, Murray, and Schmiedeler is extended to yield a spherical hoop methodology. The result is a useful tool to determine if a given spherical RR dyad will guide a moving body through a set of prescribed orientations in the desired order. Finally, we demonstrate the utility of the hoop method for order analysis of spherical RR dyads in two case studies. [DOI: 10.1115/1.4004898]

Keywords: orientation order analysis, spherical rigid-body guidance, and spherical mechanism analysis

Introduction

The objective of this effort is to devise a methodology to determine whether a given prescribed fixed axis of a spherical RR dyad guides a rigid-body through n desired spherical orientations in the desired order (1, 2, 3, 4, …). The motivation derives from commonly encountered kinematic dimensional synthesis challenges. A common mechanism or linkage design objective is the dimensional synthesis to solve rigid-body guidance tasks [1]; also known as motion generation tasks [2]. In such cases the objective is to determine the geometric parameters that define the mechanism such that the mechanism guides a moving body through a sequence of finitely separated locations in a desired order. Often kinematic synthesis algorithms for motion generation tasks yield large sets of candidate solutions that must be analyzed for motion defects; see *SYNTHETICA* by Su et al. [3], *LINCAGES* by Erdman [4], *SPHINX* by Larochelle et al. [5], and *SPHINX-PC* by Ruth and McCarthy [6]. Common motion defects encountered when using such software tools include order, circuit, and branch defects [7]. In a one degree of freedom mechanism, such as planar and spherical four-bars, the mechanism suffers from an order defect when unidirectional motion of the input or driving link does not guide the moving body through the prescribed locations in the desired order. Here, we present a novel hoop method for analyzing spherical RR dyads and four-bar mechanisms for order defects. The hoop method [8] is derived from the propeller method; an earlier work done on the order analysis of planar mechanisms by Myszka et al. [9,10].

Subsequent to Chase and Mirth's work on defining and classifying motion defects [7], Balli and Chand [11] provided a comprehensive overview of linkage defects i.e., order, branch, and circuit defects. Filemon [12] presents graphical construction techniques for four position synthesis that identify regions of the design solution space that are free from order and circuit defects. Moreover, Filemon showed that in the case of four-bar mechanisms order defects are directly related to the driving (or input) dyad and are independent of the driven (or output) dyad. Hence the spherical RR dyad order analysis methodology presented here is directly applicable to the order analysis of spherical four-bar mechanisms. Additional works examining the order analysis for both planar and spherical mechanisms can be found in Refs. [13–19].

Recently Myszka et al. [9,10] presented a new approach to determine whether a planar RR dyad will guide a moving body through a set of finitely separated positions in the desired order. Their methodology, the propeller method, utilizes relative crank angles and relative displacement poles to check the order of finitely separated planar locations associated with a prescribed fixed pivot **G** of a planar RR dyad. Here their work on order analysis of planar mechanisms is extended to spherical mechanisms, and we introduce the hoop method. For a given set of finitely separated spherical orientations and a spherical RR dyad with fixed axis **G** the hoop method utilizes the relative displacement axes of rotation to analyze the dyad for order defects.

Spherical Four-Bar Mechanisms

A four-bar mechanism consists of four links connected by four revolute joints. The links of the spherical four-bar mechanism can be represented by arcs of great circles. The links are connected to each other by revolute joints whose four axes intersect in a unique point; the center of the sphere of motion or design sphere. The links of the mechanism undergo spherical motion because the four joint axes intersect in a unique point [20]. A rigid-body undergoing spherical motion has three degrees of freedom; rotations about three mutually orthogonal axes passing through the center of the sphere. These three rotation angles define the orientation of the body with respect to the fixed coordinate frame with origin at the center of the design sphere.

A spherical four-bar mechanism as shown in Fig. 1 can be viewed as a combination of two spherical RR dyads, a driving dyad and a driven dyad. The driving fixed joint axis **O** and the driving moving axis **A** constitute the driving dyad whereas the driven fixed joint axis **C** and the driven moving axis **B** comprise the driven dyad. The link lengths of the mechanism are the driving link α, the driven link β, the coupler η, and the ground or fixed link γ, respectively. Often the two RR dyads are synthesized separately and then their floating links are joined to form a coupler to yield a closed spherical kinematic chain.

The Hoop Method

Here, we extend the planar propeller method of Myszkaet al. [9,10] to spherical RR dyads. The hoop method is a novel methodology for orientation order analysis for spherical RR dyads performing rigid-body guidance. The methodology utilizes relative crank angles and relative displacement axes of rotation to check the order of finitely separated planar locations associated with a prescribed fixed axis **G** of a spherical RR dyad. Note that this axis intersects the design sphere in two points, see Fig. 2. In the hoop method this ambiguity is resolved by utilizing the intersection that correlates to the physical location of the joint that connects the fixed and moving links.

The derivation of the hoop method is a spherical generalization of the derivation of the propeller method. The major steps in the derivation will be summarized here and the spherical nuances will be discussed in detail. Just as in the case of the planar propeller theorem and methodology, the spherical hoop method is derived from fundamental properties of crank rotation angles and relative axes of rotation. Moreover, McCarthy's spherical generalization of the center-point theorem, called the center-axis theorem [21], is utilized to yield necessary and sufficient conditions on the locations of the axes of rotation.

The hoop theorem and methodology are derived from fundamental properties of crank rotation angles and relative rotation axes. For a dyad to guide a body through the orientations in the desired order (1, 2, 3, 4, …) the relative crank angles β_{ij} must

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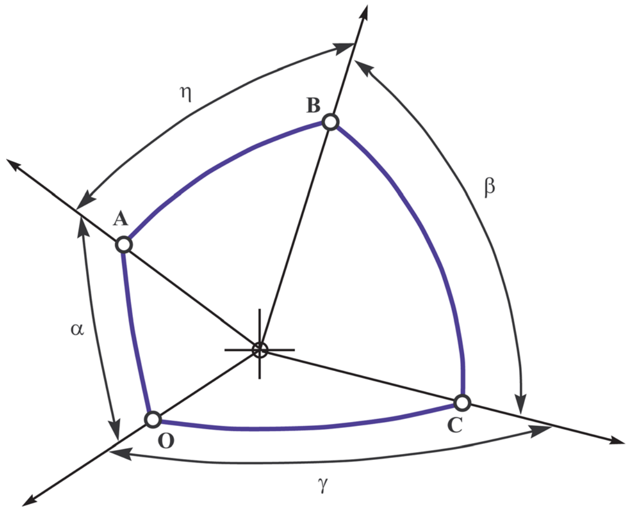

Fig. 1 Spherical four-bar mechanism

monotonically increase where β_{ij} is the change in crank angle from location i to location j. Note that without loss of generality, we regard counter-clockwise crank rotations to be positive so that $0 \le \beta_{ij} \le 2\pi$, $\forall i, j < n$, and i < j. Therefore, a dyad does not exhibit an order defect if

$$0 < \beta_{12} < \beta_{13} < \cdots < \beta_{1n} < 2\pi \tag{1}$$

Next, the center-axis theorem of McCarthy [21] expresses the geometric relationships between the fixed axis **G**, the relative rotation axis $\mathbf{S_{ij}}$, and the relative crank angle β_{ij} for a spherical RR dyad that guides a moving body through orientations $\mathbf{M_i}$, $\mathbf{M_j}$, and $\mathbf{M_k}$. The center-axis theorem states that $\angle \mathbf{S_{ij}GS_{jk}} = \frac{\beta_{ik}}{2}$ or $\frac{\beta_{ik}}{2} + \pi$. Using the center-axis theorem n necessary and sufficient conditions on the locations of the relative axes of rotation are derived per the planar case presented in Ref. [10].

$$\begin{aligned}
&0 < \angle\overline{\mathbf{GS_{12}GS_{13}}} < \angle\overline{\mathbf{GS_{12}GS_{14}}} < \cdots < \angle\overline{\mathbf{GS_{12}GS_{1n}}} < \pi\\
&0 < \angle\overline{\mathbf{GS_{12}GS_{23}}} < \angle\overline{\mathbf{GS_{12}GS_{24}}} < \cdots < \angle\overline{\mathbf{GS_{12}GS_{2n}}} < \pi\\
&0 < \angle\overline{\mathbf{GS_{13}GS_{23}}} < \angle\overline{\mathbf{GS_{13}GS_{34}}} < \cdots < \angle\overline{\mathbf{GS_{13}GS_{3n}}} < \pi\\
&\quad\vdots\\
&0 < \angle\overline{\mathbf{GS_{1n}GS_{2n}}} < \angle\overline{\mathbf{GS_{1n}GS_{3n}}} < \cdots < \angle\overline{\mathbf{GS_{1n}GS_{n-1,n}}} < \pi
\end{aligned} \tag{2}$$

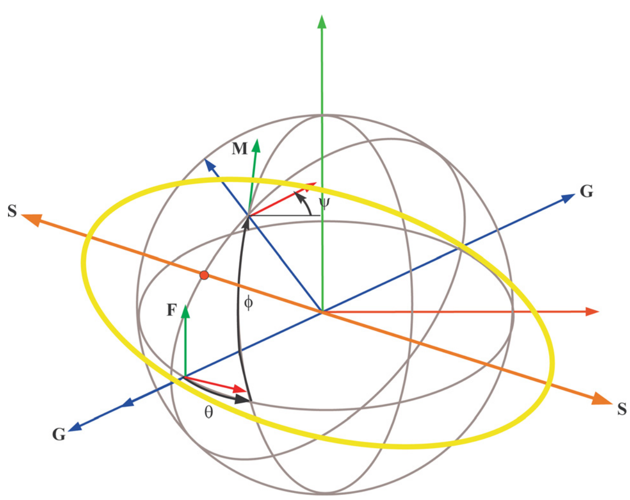

Fig. 2 The hoop, fixed axis G, rotation axis S, and the design sphere

Equation (2) states that a spherical RR dyad will guide a body through the n orientations in numerical order if for all k, starting with $\mathbf{S_{ik}}$, the hoop intersects every relative rotation axis involving orientation k in ascending order. This result is summarized in the following hoop theorem.

Theorem 1. *A spherical RR dyad will guide the moving body through orientations* 1 *to* n*, in numerical sequence, if when the hoop, initially passing through* $\mathbf{S_{12}}$*, is rotated 180 deg about the fixed axis* **G** *(in the same direction that the dyad is driven in) it intersects the relative axes of rotation in the order* $\mathbf{S_{13}} \Rightarrow \mathbf{S_{14}} \Rightarrow \ldots \Rightarrow \mathbf{S_{1n}} \Rightarrow \mathbf{S_{2n}}$.

Corollary. *If the hoop does not intersect the rotation axes in the order* $S_{13} \Rightarrow S_{14} \Rightarrow \ldots \Rightarrow S_{1n} \Rightarrow S_{2n}$ *then the dyad will not guide the moving body through the orientations in their numerical sequence.*

For a given spherical RR dyad with fixed axis **G** and a set of n desired orientations the hoop method is implemented as follows. First a great circle on the design sphere, referred to as the hoop, is defined. The hoop initially passes through the fixed axis **G** and the relative rotation axis $\mathbf{S_{12}}$ and is then rotated in a counter-clockwise or clockwise direction (corresponding to the direction of motion of the physical dyad) for π radians. The order in which the dyad guides the moving body through the desired orientations is established by tracking the sequence in which the hoop intersects the relative axes of rotation associated with the n desired orientations. Note that the hoop will intersect the relative rotation axes on both sides of the design sphere simultaneously. Furthermore, for the dyad to guide the moving body through four orientations in numerical sequence the hoop must intersect the relative rotation axes in the sequence $\mathbf{S_{13}} \Rightarrow \mathbf{S_{14}} \Rightarrow \mathbf{S_{24}}$. Moreover, for five orientations the sequence must be $\mathbf{S_{13}} \Rightarrow \mathbf{S_{14}} \Rightarrow \mathbf{S_{15}} \Rightarrow \mathbf{S_{25}}$. Note that in Figs. 4 and 6 the hoop is shown in yellow and in its initial orientation.

As an alternative to visually representing the hoop to determine the order in which it intersects the relative rotation axes an analytic formulation of the relevant spherical trigonometry may be used. Consider the relative rotation axis $\mathbf{S_{ij}}$. The angle that the hoop must be rotated to intersect this rotation axis is equal to the interior angle at the vertex **G** of the spherical triangle $\mathbf{GS_{12}S_{ij}}$. Hence the order that the hoop intersects the relative rotation axes may be determined by computing each individual hoop rotation angle. The method proceeds as follows:

(1) Determine the direction of hoop rotation.
(2) Let **n** be a unit normal to the plane spanned by **G** and $\mathbf{S_{12}}$ in the direction determined in step 1.

$$\mathbf{n} = \frac{\mathbf{G} \times \mathbf{S_{12}}}{\|\mathbf{G} \times \mathbf{S_{12}}\|} \text{ or } \frac{\mathbf{S_{12}} \times \mathbf{G}}{\|\mathbf{G} \times \mathbf{S_{12}}\|}$$

(3) Let δ_{ij} be the interior angle at **G** of the spherical triangle $\mathbf{GS_{12}S_{ij}}$ where $\mathbf{S_{ij}} \cdot \mathbf{n} \ge 0$. If $\mathbf{S_{ij}} \cdot \mathbf{n} < 0$ then let $\mathbf{S_{ij}} = -\mathbf{S_{ij}}$. This assures that all rotation axes intersections are analyzed in the hemisphere associated with **n**.
(4) Using the spherical law of cosines [20] determine δ_{ij}.

$$\cos(\delta_{ij}) = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}$$

where $\cos(a) = \mathbf{G} \cdot \mathbf{S_{12}}$, $\cos(b) = \mathbf{G} \cdot \mathbf{S_{ij}}$, and $\cos(c) = \mathbf{S_{ij}} \cdot \mathbf{S_{12}}$.
(5) The angles δ_{ij} reveal the order in which the hoop intersects the relative rotation axes.

Hoop Method Implementation

Given a spherical RR dyad that guides a rigid-body through n orientations the task is to determine if the dyad guides the body through the orientations in the desired order (i.e., 1, 2, 3, 4,....) or not. First, determine the $\binom{n}{2}$ (i.e., n choose 2) relative rotation

Table 1 Five prescribed orientations

Nos.	x_0	x_1	x_2	X_3
M_1	1.0	0.0	0.0	0.0
M_2	0.37721	0.82336	0.38967	0.16722
M_3	0.0078934	0.041131	0.085164	−0.99549
M_4	0.039457	0.77456	−0.60494	−0.18041
M_5	−0.30301	−0.36492	0.85697	0.20157

axes associated with the n orientations. Next, generate the design sphere with fixed axis **G**, the relative rotation axes, and the hoop in its initial orientation. Proceed by rotating the hoop about **G** in the same direction that the dyad is to be actuated and record the sequence in which the hoop intersects the relative rotation axes or use the spherical trigonometric formulation to analytically determine the intersection sequence. Finally, the hoop theorem is used to analyze the sequence to determine if the orientations will be reached in the desired order.

Case Study #1

Consider five orientations defined by the four Euler parameters x_1, x_2, x_3, and x_4 listed in Table 1 and the fixed axis $\mathbf{G} = (0.054261, -0.996977, \text{ and } 0.055603)$ of a spherical RR dyad that solves the motion generation problem presented by Brunnthaler et al. [22].

In Fig. 3 the five orientations $\mathbf{M_1}$, $\mathbf{M_2}$, $\mathbf{M_3}$, $\mathbf{M_4}$, and $\mathbf{M_5}$ and the fixed axis **G** are shown on the design sphere. The next step is to determine the 10 relative rotation axes **S** associated with the spherical orientations. The hoop, a great circle on the design sphere passing through **G** and $\mathbf{S_{12}}$ is generated, see Fig. 4. Next, as the hoop is rotated the sequence in which it intersects the relative rotation axes is noted.

As the hoop is rotated in counter-clockwise turn of π radians about the fixed axis **G**, starting from the rotation axis $\mathbf{S_{12}}$, it intersects the remaining rotation axes in the order $\mathbf{S_{23}}(25.9) \Rightarrow \mathbf{S_{45}}(45.4) \Rightarrow \mathbf{S_{25}}(74.1) \Rightarrow \mathbf{S_{13}}(80.2) \Rightarrow \mathbf{S_{24}}(96.1) \Rightarrow \mathbf{S_{15}}(129.4) \Rightarrow \mathbf{S_{14}}(151.3) \Rightarrow \mathbf{S_{35}}(154.1) \Rightarrow \mathbf{S_{34}}(175.7)$. The hoop rotation angles (degrees), determined analytically for each intersection, have been included in parentheses. The hoop theorem requires that the rotation axes must be intersected in the following order: $\mathbf{S_{13}} \Rightarrow \mathbf{S_{14}} \Rightarrow \mathbf{S_{15}} \Rightarrow \mathbf{S_{25}}$. Note that the hoop's intersections with the other five rotation axes are irrelevant when applying the hoop

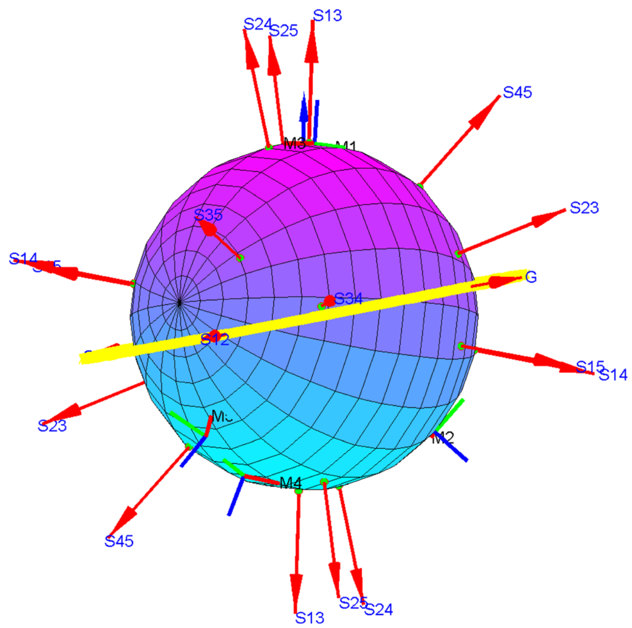

Fig. 4 The hoop through fixed axis G and the rotation axis S_{12}

theorem for five orientations. The order indicates, per the hoop theorem, that this RR dyad will not guide the moving body through the spherical orientations in their numerical sequence. Hence this dyad suffers from an order defect.

Case Study #2

Here, we again consider the same five orientation problem with another solution spherical dyad presented in Ref. [22]. The fixed axis $\mathbf{G} = (-0.349442, -0.144163, \text{ and } 0.925801)$ of another spherical RR dyad that solves the motion generation problem is now analyzed. We proceed as before and apply the hoop methodology. In Fig. 5 the five orientations, the fixed axis **G**, and the relative rotation axes are shown on the design sphere.

As the hoop is rotated clockwise turn of π radians about **G**, starting from $\mathbf{S_{12}}$ (see Fig. 6), it intersects the remaining rotation axes in the order $\mathbf{S_{35}}(1.30) \Rightarrow \mathbf{S_{13}}(14.4) \Rightarrow \mathbf{S_{45}}(51.4) \Rightarrow \mathbf{S_{14}}(68.9) \Rightarrow \mathbf{S_{23}}(83.5) \Rightarrow \mathbf{S_{15}}(97.0) \Rightarrow \mathbf{S_{24}}(134.4) \Rightarrow \mathbf{S_{34}}(152.5) \Rightarrow$

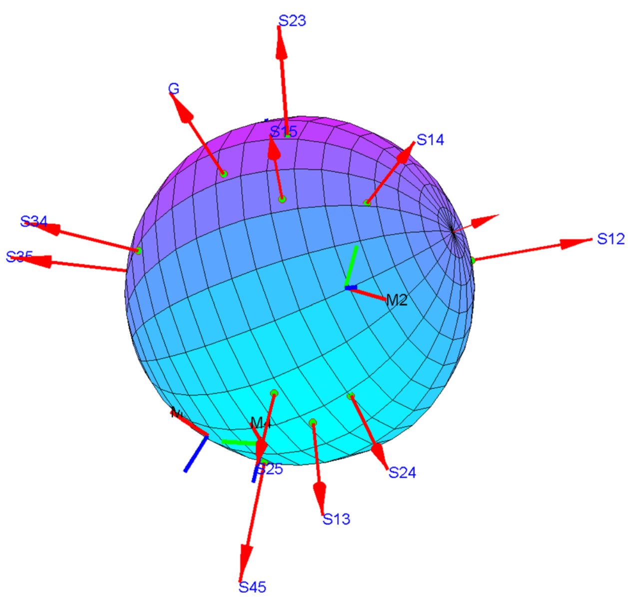

Fig. 3 Five spherical orientations with the fixed axis G

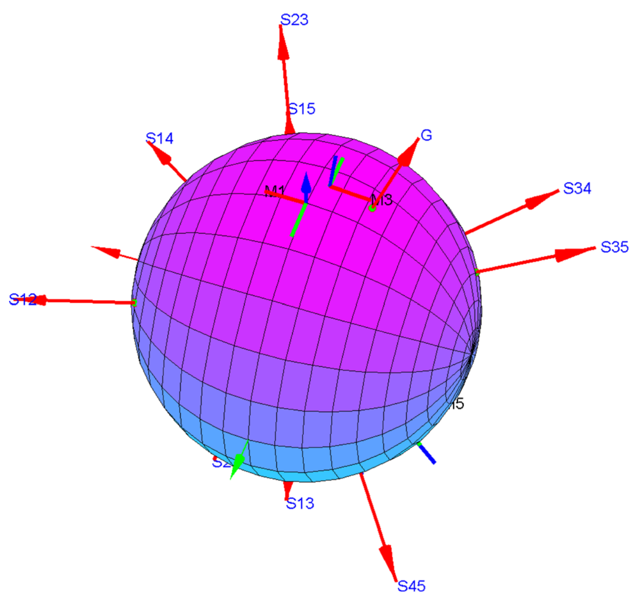

Fig. 5 Five orientations with the fixed axis G

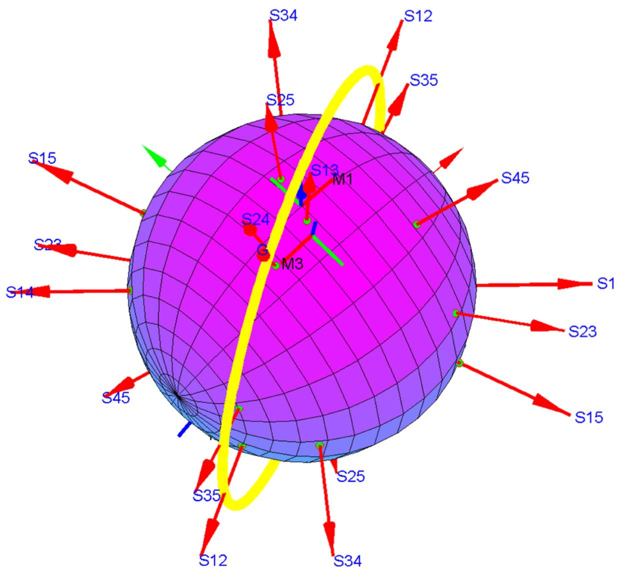

Fig. 6 The hoop on the design sphere

$\mathbf{S}_{25}(162.5)$. The hoop rotation angles (degrees), determined analytically for each intersection, are included in parentheses. The order indicates, per the hoop theorem, which this RR dyad will guide the moving body through the spherical orientations in their numerical sequence. Hence this dyad does not suffer from an order defect.

Conclusions

A novel methodology for orientation order analysis of spherical RR dyads has been presented. The prior work on the order analysis of planar RR dyads via the propeller method was extended to yield a spherical hoop methodology. The hoop is a great circle on the design sphere that intersects the fixed axis of the spherical RR dyad. The hoop method involves rotating the hoop about the fixed axis and noting the order in which the relative rotation axes are encountered. This method was shown to be useful for performing the order analysis of spherical RR dyads. Finally, the utility of the hoop method was demonstrated in two case studies.

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